MATH 312
Lecture 3: Metamorphosis

ENTER THE $\begin{bmatrix} M & A & T \\ R & I & X \end{bmatrix}$

- Consider the linear system

$$\boxed{\begin{aligned} 3x - 4y + 5z &= 10 \\ 4x + y - z &= 12 \\ 2x + 2y + 3z &= 1 \end{aligned}}$$

- Let's COMPRESS the notation to make our lives easier: We use COLUMNS for variables and ROWS for equations

$$\text{3 rows}\left\{\begin{array}{c} \\ \text{Eqn 1} \\ \text{Eqn 2} \\ \text{Eqn 3} \end{array}\right. \underbrace{\begin{array}{cccc} x & y & z & \text{(Right side)} \\ \left[\begin{array}{cccc} 3 & -4 & 5 & 10 \\ 4 & 1 & -1 & 12 \\ 2 & 2 & 3 & 1 \end{array}\right] \end{array}}_{\text{4 columns}}$$

The text calls this an "augmented" matrix

- Now, the usual Gaussian Elimination "moves" become ROW OPERATIONS on this matrix.
  - Subtract $\frac{4}{3}R1$ from $R2$, i.e. $R2' = R2 - \frac{4}{3}R1$

and so forth. This is JUST NOTATION right now, nothing more!

In *matrix notation*, Gaussian Elimination looks like:

$$\begin{bmatrix} 3 & -4 & 5 & 10 \\ 4 & 1 & -1 & 12 \\ 2 & 2 & 3 & 1 \end{bmatrix} \xrightarrow[\substack{R3' = R3 - \frac{2}{3}R1 \\ \text{(clean out } x)}]{R2' = R2 - \frac{4}{3}R1} \begin{bmatrix} 3 & -4 & 5 & 10 \\ 0 & 7/3 & -8/3 & 26/3 \\ 0 & 14/3 & -1/3 & -7/3 \end{bmatrix}$$

$R3' = R3 - 2R2$ (clean out y)

$$\begin{bmatrix} 3 & -4 & 5 & 10 \\ 0 & 7/3 & -8/3 & 26/3 \\ 0 & 0 & 5 & -59/3 \end{bmatrix}$$

Triangular, Hallelujah!!

substitute "upwards"

$$\boxed{\begin{aligned} z &= -\frac{59}{15} \\ y &= \_\_\_ \\ x &= \_\_\_ \end{aligned}}$$

- More general problem: changing the RIGHT SIDE

Let's start with a 2×2 system:

$$\begin{aligned} 3x + 2y &= 5 \\ 6x - 5y &= 1. \end{aligned}$$

It is "easy" to check that the (ONLY) solution is $x = 1$, $y = 1$. But what if we change the right side:

$$\begin{aligned} 3x + 2y &= 4 \\ 6x - 5y &= 1 \end{aligned}$$

Now what?

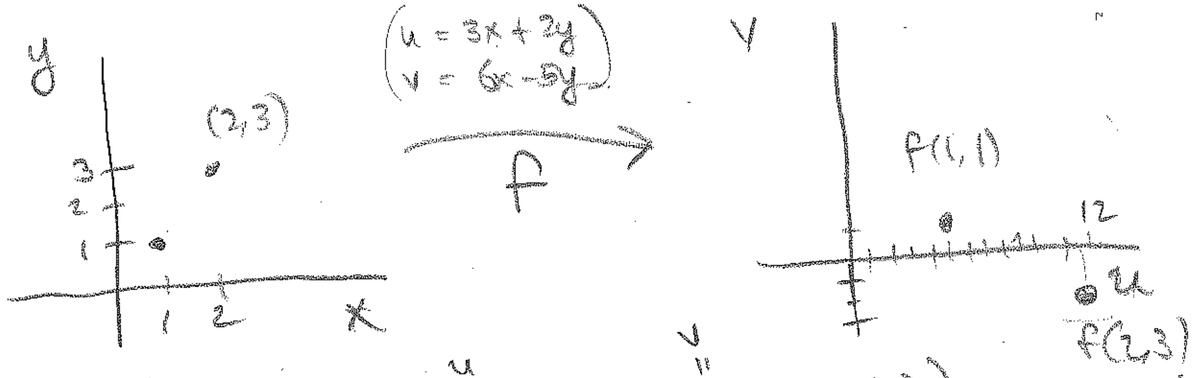
$\boxed{\text{Question}}$?

CAN WE SOLVE THIS SYSTEM FOR <u>ALL</u> POSSIBLE CHOICES OF "RIGHT SIDE"?

- So, let's replace the right side by "unknowns" (u & v):

$$\boxed{\begin{aligned} 3x + 2y &= u \\ 6x - 5y &= v \end{aligned}} \qquad \text{Matrix} = \begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$$

- What we <u>WANT</u> is a <u>FORMULA</u> for x and y in terms of u and v. But first, note that we have a FUNCTION f taking input variables (x,y) to output variables (u,v).

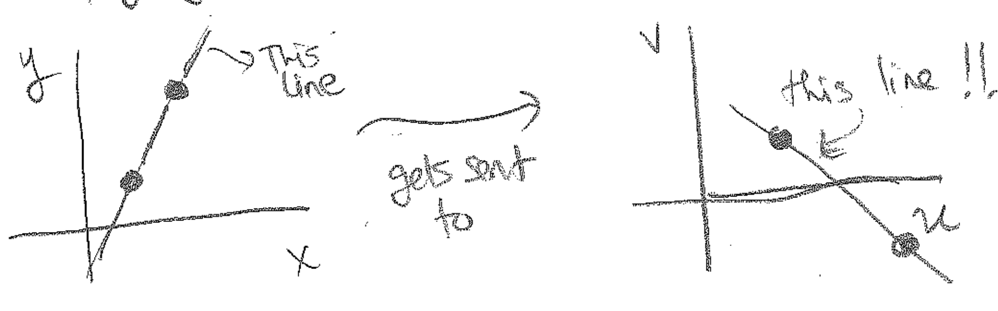
- 

eg: $f(\overset{x}{2},\overset{y}{3}) = (\overset{u}{3(2) + 2(3)}, \overset{v}{6(2) - 5(3)})$
$= (12, -3)$

$f(1,1) = (5,1)$, and so forth..

- f is <u>LINEAR</u>: given any scalar a,

$$\left[\begin{aligned} &f(ax, ay) = (au, av) = a\, f(x,y) \\ \text{and: } &f(x+x', y+y') = f(x,y) + f(x',y') \end{aligned}\right]$$



- Again,

$$\boxed{\begin{aligned} 3x + 2y &= u \\ 6x - 5y &= v \end{aligned}} \qquad \text{Matrix} = \begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$$

If we have a new linear equation system which takes u and v as variables, i.e.,

$$\boxed{\begin{aligned} 2u + v &= s \\ u - 2v &= t, \end{aligned}} \qquad \text{Matrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

- How do s and t relate to x and y?

Well,

$$\underbrace{2u}_{= 2(3x+2y)} + \underbrace{v}_{= (6x-5y)} = s$$

So

Similarly,

$$\boxed{\begin{aligned} 12x - y &= s \\ -9x + 12y &= t \end{aligned}} \qquad \text{Matrix} = \begin{bmatrix} 12 & -1 \\ -9 & 12 \end{bmatrix}$$

We <u>DEFINE</u> Matrix multiplication so that the product of $\begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ gives us $\begin{bmatrix} 12 & -1 \\ -9 & 8 \end{bmatrix}$. So,

(dot product)

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix} = \begin{bmatrix} 2 \times 3 + 1 \times 6 & 2 \times 2 + 1 \times -5 \\ 1 \times 3 - 2 \times 6 & 1 \times 2 + -2 \times -5 \end{bmatrix}$$

The recipe is:

$$\boxed{\text{product}_{ij} = \sum_{k=1}^{\#\text{cols(left)} = \#\text{rows(right)}} \text{left}_{ik} \times \text{right}_{kj}}$$

- WARNING We don't get $AB = BA$ in general!
- A very special matrix is the IDENTITY: $\begin{bmatrix}1&0\\0&1\end{bmatrix}$. It has the property that $A \times Id = A = Id \times A$ for any choice of matrix A.

Remember Our goal was to SOLVE for (x,y) in

$$\underbrace{\begin{bmatrix}3&2\\6&-5\end{bmatrix}}_{= A}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}u\\v\end{bmatrix}.$$

If there was an INVERSE for A, i.e. a matrix A^{-1} which satisfied $A^{-1}A = Id$, then we get

$$A^{-1}A\begin{bmatrix}x\\y\end{bmatrix} = A^{-1}\begin{bmatrix}u\\v\end{bmatrix}$$

$$\text{or } \underbrace{Id\begin{bmatrix}x\\y\end{bmatrix}}_{= \begin{bmatrix}x\\y\end{bmatrix}} = A^{-1}\begin{bmatrix}u\\v\end{bmatrix},$$

and we'd be done